

Realizations of $su(1, 1)$ and $U_q(su(1, 1))$ and generating functions for orthogonal polynomials

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Abstract: Positive discrete series representations of the Lie algebra $su(1, 1)$ and the quantum algebra $U_q(su(1, 1))$ are considered. The diagonalization of a self-adjoint operator (the Hamiltonian) in these representations and in tensor products of such representations is determined, and the generalized eigenvectors are constructed in terms of orthogonal polynomials. Using simple realizations of $su(1, 1)$, $U_q(su(1, 1))$, and their representations, these generalized eigenvectors are shown to coincide with generating functions for orthogonal polynomials. The relations valid in the tensor product representations then give rise to new generating functions for orthogonal polynomials, or to Poisson kernels. In particular, a group theoretical derivation of the Poisson kernel for Meixner-Pollaczak and Al-Salam–Chihara polynomials is obtained.

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I Introduction

The Lie algebra $su(1,1)$ (or $so(2,1)$) has been extensively used as spectrum generating algebra in many simple quantum systems, such as the non-relativistic Coulomb problem, the isotropic harmonic oscillator, Schrödinger's relativistic equation, and the Dirac-Coulomb problem [1, and references therein]. In some interacting boson models the Hamiltonian H can be written as a linear combination of the $su(1,1)$ basis elements J_0 , J_{\pm} , such that $H^{\dagger} = H$, see e.g. the superfluid bose system [2, 3]. In such boson models, the representations that play a role are the positive discrete series representations of $su(1,1)$, often denoted by $\mathcal{D}^+(k)$. For a given Hamiltonian $H = aJ_0 + bJ_+ + cJ_- + d$, with $a^* = a$, $d^* = d$ and $b^* = c$, the important questions are : (1) its spectrum (continuous or discrete); (2) the expansion of its eigenvectors in terms of the boson states $|k, n\rangle$ (corresponding to eigenstates of J_0 in the representations $\mathcal{D}^+(k)$).

In [4] a particular element, $H = 2J_0 - J_+ - J_-$, was chosen, and it was shown that the expansion coefficients of the eigenstates of H in the basis $|k, n\rangle$ of the representation $\mathcal{D}^+(k)$ are given as generalized Laguerre polynomials (see also [5]). The main purpose of [4] was then to consider tensor product representations, and deduce in a Lie algebraic way addition or convolution formulas for orthogonal polynomials (in this case for generalized Laguerre and Jacobi polynomials, cfr. [4, (1.1)]). Following this, the most general Hamiltonian with real coefficients (up to an overall constant and scaling factor) $H = \sigma J_0 - J_+ - J_-$ ($\sigma \in \mathbb{R}$) was considered in [6], and it was shown that the spectrum of H depends upon whether $|\sigma| = 2$, $|\sigma| < 2$ or $|\sigma| > 2$ (see also [7] for a particular representation). In each of these cases, the expansion coefficients of the Hamiltonian eigenstates in the $|k, n\rangle$ -basis are orthogonal polynomials. In section II, the main results of [4, 6] are summarized in the present framework of eigenstates (generalized eigenvectors).

Although the physical applications or interpretations are clear from the above observations, this paper is primarily dealing with a number of mathematical consequences of choosing particular realizations for the positive discrete series representations.

The general framework for the positive discrete series representations is outlined in section II. The three type of operators are given, together with their spectrum and expansion coefficients (orthogonal polynomials). The tensor product of two positive discrete series representations is considered, and just as there are “coupled” and “uncoupled” eigenvectors for J_0 in the tensor product, one can also define coupled and uncoupled eigenstates of the Hamiltonian. The expansion coefficients of uncoupled eigenstates into coupled eigenstates (in an irreducible component of the tensor product), cfr. equation (2.13), are again orthogonal polynomials (Jacobi, continuous Hahn, or Hahn polynomials).

All the relations given in section II (deduced in [4, 6]) are obtained by Lie algebraic methods only. The purpose of the present paper is to examine the implications of these relations once a realization for the Lie algebra $su(1,1)$ and its representations $\mathcal{D}^+(k)$ are chosen. For $su(1,1)$, there exists a classical realization such that the J_0 -diagonal basis states $|k, n\rangle$ of $\mathcal{D}^+(k)$ are simply the monomials z^n . This realization is considered in section III. As a consequence, the eigenstates of the Hamiltonian H simply become generating functions for the orthogonal polynomials (i.e. the expansion coefficients), and also the J_0 -diagonal basis states in an irreducible component of the tensor product become a simple functions. Then, it remains to consider the relation connecting coupled and uncoupled eigenstates of

the Hamiltonian in the tensor product, and investigate its explicit form in this realization. For the case of Laguerre and Jacobi polynomials, this is done in section IV; the outcome is a generalization of classical generating functions for Laguerre polynomials. For the case of Meixner and Hahn polynomials, this analysis is performed in section V; the outcome is the Poisson kernel formula for Meixner (or Meixner-Pollaczek) polynomials, equation (5.5). Although the formula thus obtained is known, the method used to reach it is original and interesting : in this Lie algebraic framework it is a simple consequence of a very general expansion (equation (2.13)) in a particular realization.

Then we turn our attention to the q -analog of the results obtained so far. Since the method is purely algebraic, it works not only for the Lie algebra $su(1,1)$ but also for the quantum algebra $U_q(su(1,1))$ [8]. In section VI, the quantum algebra $U_q(su(1,1))$ is defined, and the positive discrete series representations with a standard basis are given. Next, a self-adjoint operator playing the role of Hamiltonian is considered, and its eigenstates are expanded in the standard basis with expansion coefficients proportional to Al-Salam–Chihara polynomials. Just as for the $su(1,1)$ case, coupled and uncoupled eigenstates of this operator in the tensor product are constructed and the expansion of one into the other is given in terms of Askey-Wilson polynomials. The results presented in section VI are mainly taken from [6]. Just as for $su(1,1)$, the purpose is now to investigate the implication of these formulas once a realization is chosen. For $U_q(su(1,1))$, this realization is given in section VII, and the standard basis for positive discrete series representations consist again of monomials z^n . Finally, the relation connecting coupled and uncoupled eigenstates is shown to give rise to the (symmetric) Poisson kernel formula for Al-Salam–Chihara polynomials in this realization. Although this formula has recently been found by means of classical methods, the present deduction is simple, purely algebraic, and a natural q -analog of the Poisson kernel for Meixner-Pollaczek polynomials in the $su(1,1)$ framework.

In a final section, two different realizations of $su(1,1)$ and its representations are considered. These realizations have been used in physical models. We show again how our general results of section II have interesting implications for special functions. In particular, integrals over products of Laguerre and Hermite polynomials are expressed in terms of a Meixner polynomial. This section is completed by some general comments and further outlook.

II The Lie algebra $su(1,1)$ and positive discrete series representations

The Lie algebra $su(1,1)$ is generated by J_0, J_{\pm} subject to the relations

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2J_0, \quad (2.1)$$

with the conditions $J_0^{\dagger} = J_0$ and $J_{\pm}^{\dagger} = J_{\mp}$. The positive discrete representations [9] $\mathcal{D}^+(k)$ are labeled by a positive real number k . The representation space is $\ell^2(\mathbb{Z}_+)$, with orthonormal basis vectors denoted by $e_n^{(k)}$, with $n = 0, 1, 2, \dots$ (and sometimes denoted by k, n). The

explicit action of the generators in this representation (k) is given by :

$$\begin{aligned} J_0 e_n^{(k)} &= (n+k) e_n^{(k)}, \\ J_+ e_n^{(k)} &= \sqrt{(n+1)(2k+n)} e_{n+1}^{(k)} \\ J_- e_n^{(k)} &= \sqrt{n(2k+n-1)} e_{n-1}^{(k)}. \end{aligned} \quad (2.2)$$

In two previous papers [4, 6], a recurrence operator in the Lie algebra $su(1,1)$ was related to a Jacobi matrix. Rather than working with the spectral theory of Jacobi matrices, it will be more appropriate for this paper to use the (equivalent) notion of formal or generalized eigenvectors, to be identified with eigenstates of a Hamiltonian. In [6], the formal eigenvectors

$$v^{(k)}(x) = \sum_{n=0}^{\infty} l_n^{(k)}(x) e_n^{(k)} \quad (2.3)$$

of the self-adjoint operator $X = \sigma J_0 - J_+ - J_-$ ($\sigma \in \mathbb{R}$) in the representation (k) were studied; in a boson model such as referred to in the introduction the Hamiltonian H is essentially equal to this operator X . The formal vector $v^{(k)}(x)$ is a generalized eigenvector of X for the eigenvalue $\lambda(x)$ provided $l_n^{(k)}(x)$ satisfies a three-term recurrence relation [4, 6]. This leads to an identification of $l_n^{(k)}(x)$ with orthogonal polynomials in x . These polynomials associated with X are of different type for $|\sigma| = 2$, $|\sigma| < 2$ or $|\sigma| > 2$. Labelling the operator in the three distinct cases as follows :

$$X_2 = 2J_0 - J_+ - J_-, \quad (2.4)$$

$$X_\phi = -2 \cos \phi J_0 + J_+ + J_-, \quad 0 < \phi < \pi, \quad (2.5)$$

$$X_c = -(c+1/c)J_0 + J_+ + J_-, \quad 0 < c < 1, \quad (2.6)$$

we can give a summary of some results obtained earlier in Table 1. The spectrum of X is determined by the support of the measure of the associated orthogonal polynomials, being continuous for $|\sigma| = 2$ and $|\sigma| < 2$, and discrete for $|\sigma| > 2$.

Table 1: Orthogonal polynomials appearing in the formal eigenvector $v^{(k)}(x)$.

X	$l_n^{(k)}(x)$	polynomial	eigenvalue $\lambda(x)$	spectrum
X_2	$\sqrt{\frac{n!}{(2k)_n}} L_n^{(2k-1)}(x)$	generalized Laguerre	x	$0 \leq x < \infty$
X_ϕ	$\sqrt{\frac{n!}{\Gamma(2k+n)}} P_n^{(k)}(x; \phi)$	Meixner-Pollaczek	$2x \sin \phi$	$x \in \mathbb{R}$
X_c	$\sqrt{\frac{(2k)_n}{n!}} c^n M_n(x; 2k; c^2)$	Meixner	$(c-1/c)(k+x)$	$x \in \mathbb{N}$

In Table 1, $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ is the common notation for the Pochhammer symbol in terms of the classical Γ function. The definition of the orthogonal polynomials in terms

of hypergeometric series is as follows [10] :

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1 \left[\begin{matrix} -n \\ \alpha+1 \end{matrix}; x \right], \quad (\alpha > -1); \quad (2.7)$$

$$P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1 \left[\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix}; 1 - e^{-2i\phi} \right], \quad (\lambda > 0); \quad (2.8)$$

$$M_n(x; \beta; c) = {}_2F_1 \left[\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - 1/c \right], \quad (\beta > 0). \quad (2.9)$$

The notation for hypergeometric series is the standard one [11, 12].

By considering the tensor product decomposition for $su(1, 1)$ representations, new summation formulas or convolution theorems were obtained for these polynomials. The tensor product decomposes as follows [9] :

$$(k_1) \otimes (k_2) = \bigoplus_{j=0}^{\infty} (k_1 + k_2 + j). \quad (2.10)$$

The “coupled basis vectors” are written in terms of the uncoupled ones by means of the Clebsch-Gordan coefficients :

$$e_n^{(k_1 k_2)k} = \sum_{n_1, n_2} C_{n_1, n_2, n}^{k_1, k_2, k} e_{n_1}^{(k_1)} \otimes e_{n_2}^{(k_2)}. \quad (2.11)$$

Herein, $k = k_1 + k_2 + j$ for some integer $j \geq 0$, and the sum is such that $n_1 + n_2 = j + n$. Explicit expression for the Clebsch-Gordan coefficients are given, e.g. in Ref. [4].

In the tensor product space, the generalized eigenvectors of $\Delta(X) = X \otimes 1 + 1 \otimes X$ can be considered. Again, there are “uncoupled” eigenvectors $v^{(k_1)}(x_1)v^{(k_2)}(x_2)$ with eigenvalue $\lambda(x_1 + x_2)$, but also “coupled” eigenvectors $v^{(k_1 k_2)k}(x_1 + x_2)$ defined as follows :

$$v^{(k_1 k_2)k}(x_1 + x_2) = \sum_{n=0}^{\infty} l_n^{(k)}(x_1 + x_2) e_n^{(k_1 k_2)k}, \quad (2.12)$$

where $k = k_1 + k_2 + j$ for some integer $j \geq 0$. It was shown in [4, 6] that

$$v^{(k_1)}(x_1)v^{(k_2)}(x_2) = \sum_{j=0}^{\infty} S_j^{(k_1, k_2)}(x_1, x_2) v^{(k_1 k_2)k_1 + k_2 + j}(x_1 + x_2), \quad (2.13)$$

where S is again an orthogonal polynomial given in Table 2, according to which of the three cases is considered. The constants in this table are determined by

$$C_1 = (j! / ((2k_1)_j (2k_2)_j (2k_1 + 2k_2 + j - 1)_j))^{1/2}, \quad (2.14)$$

$$C_2 = (j! (2k_1 + 2k_2 + 2j - 1) \Gamma(2k_1 + 2k_2 + j - 1) / (\Gamma(2k_1 + j) \Gamma(2k_2 + j)))^{1/2}, \quad (2.15)$$

$$C_3 = C_2 (2k_1)_j / j!. \quad (2.16)$$

Table 2: Orthogonal polynomial appearing in the expansion (2.13).

X	$S_j^{(k_1, k_2)}(x_1, x_2)$	polynomial
X_2	$C_1(-1)^j(x_1 + x_2)^j P_j^{(2k_1-1, 2k_2-1)}\left(\frac{x_2-x_1}{x_2+x_1}\right)$	Jacobi
X_ϕ	$C_2(-2\sin\phi)^j p_j(x_1; k_1, k_2 - i(x_1 + x_2), k_1, k_2 + i(x_1 + x_2))$	continuous Hahn
X_c	$C_3(-c + 1/c)^j (-x_1 - x_2)_j Q_j(x_1; 2k_1 - 1, 2k_2 - 1, x_1 + x_2)$	Hahn

The polynomials appearing in Table 2 are defined as follows [10] :

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1-x}{2} \right], \quad (2.17)$$

$$p_n(x; a, b, c, d) = i^n \frac{(a+c)_n (a+d)_n}{n!} {}_3F_2 \left[\begin{matrix} -n, n + a + b + c + d - 1, a + ix \\ a + c, a + d \end{matrix}; 1 \right], \quad (2.18)$$

$$Q_n(x; a, b, N) = {}_3F_2 \left[\begin{matrix} -n, n + a + b + 1, -x \\ a + 1, -N \end{matrix}; 1 \right]. \quad (2.19)$$

A number of interesting new convolution identities for orthogonal polynomials were constructed in [6] from the relation ($k = k_1 + k_2 + j$)

$$\sum_{n_1+n_2=n+j} C_{n_1, n_2, n}^{k_1, k_2, k} l_{n_1}^{(k_1)}(x_1) l_{n_2}^{(k_2)}(x_2) = l_n^{(k)}(x_1 + x_2) S_j^{(k_1, k_2)}(x_1, x_2). \quad (2.20)$$

In the simplest case, where $l_n^{(k)}(x)$ corresponds to a generalized Laguerre polynomial and $S_j^{(k_1, k_2)}(x_1, x_2)$ to a Jacobi polynomial, this relation reduces to [4, (1.1)]. Recently, this equation has been applied to find transformation brackets for $U(N)$ boson models [13].

III Realization of $su(1, 1)$

In this section we give a realization of $su(1, 1)$ and of the discrete series representation $\mathcal{D}^+(k)$. Assume that $k > 1/2$, and consider the Hilbert space of analytic functions $f(z)$ ($z \in \mathbb{C}$) on the unit disc $|z| < 1$, with inner product [9]

$$(f_1, f_2) = \frac{2k-1}{\pi} \int \int_{|z|<1} f_1(z) \overline{f_2(z)} (1 - |z|^2)^{2k-2} dx dy, \quad (z = x + iy). \quad (3.1)$$

The following functions form an orthonormal basis :

$$e_n^{(k)} \equiv e_n^{(k)}(z) = \sqrt{\frac{(2k)_n}{n!}} z^n, \quad (3.2)$$

since $(e_n^{(k)}, e_m^{(k)}) = \delta_{m,n}$. The realization of the $su(1, 1)$ basis elements reads as follows :

$$J_0 = z \frac{d}{dz} + k, \quad J_- = \frac{d}{dz}, \quad J_+ = z^2 \frac{d}{dz} + 2kz. \quad (3.3)$$

It is easy to verify that the action of these operators on the basis (3.2) is indeed the same as in (2.2).

Next, consider the formal vectors (2.3), which are now denoted by $v^{(k)}(x, z)$ since in this realization the basis vectors become functions of z . Since the vectors $e_n^{(k)}$ are proportional to z^n , one can see that the $v^{(k)}(x, z)$ are in fact generating functions for the polynomials $l_n^{(k)}(x)$. The explicit forms of the vectors is given in Table 3, for each of the three cases consider here.

Table 3: Explicit form of the vectors $v^{(k)}(x, z)$ in the present realization.

X	series	generating function
X_2	$\sum_{n=0}^{\infty} L_n^{(2k-1)}(x) z^n$	$(1-z)^{-2k} e^{xz/(z-1)}$
X_ϕ	$\Gamma(2k)^{-1} \sum_{n=0}^{\infty} P_n^{(k)}(x; \phi) z^n$	$(1 - e^{i\phi} z)^{-k+ix} (1 - e^{-i\phi} z)^{-k-ix} / \Gamma(2k)$
X_c	$\sum_{n=0}^{\infty} \frac{(2k)_n c^n}{n!} M_n(x; 2k, c^2) z^n$	$(1 - z/c)^x (1 - cz)^{-x-2k}$

For each of the cases, the generating function can be found in Ref. [10]. One can also find it by explicitly solving the first order differential equation

$$Xv^{(k)}(x, z) = \lambda(x)v^{(k)}(x, z), \quad (3.4)$$

where X is one of (2.4)-(2.6), in the realization (3.3).

Next, consider the tensor product of two representations $(k_1) \otimes (k_2)$ in the present realization, and the coupled vector

$$e_n^{(k_1 k_2)k}(z_1, z_2) = \sum_{n_1, n_2} C_{n_1, n_2, n}^{k_1, k_2, k} e_{n_1}^{(k_1)}(z_1) e_{n_2}^{(k_2)}(z_2). \quad (3.5)$$

Using properties of the generating function for the Clebsch-Gordan coefficients, this expression can be written as follows :

$$e_n^{(k_1 k_2)k}(z_1, z_2) = \left[\frac{(2k_1)_j (2k_2)_j (2k_1 + 2k_2 + 2j)_n}{j! n! (2k_1 + 2k_2 + j - 1)_j} \right]^{1/2} \times (-z_1 + z_2)^j z_1^n {}_2F_1 \left[\begin{matrix} -n, 2k_2 + j \\ 2k_1 + 2k_2 + 2j \end{matrix}; 1 - z_2/z_1 \right], \quad (3.6)$$

where, as before, $k = k_1 + k_2 + j$.

In the following sections we shall consider the explicit forms of (2.12) and (2.13) in this realization, and its consequences.

IV The case of Jacobi polynomials

Consider equation (2.13) in the case $X = X_2$,

$$v^{(k_1)}(x_1, z_1) v^{(k_2)}(x_2, z_2) = \sum_{j=0}^{\infty} S_j^{(k_1, k_2)}(x_1, x_2) v^{(k_1 k_2)k_1 + k_2 + j}(x_1 + x_2, z_1, z_2); \quad (4.1)$$

in this situation the polynomials S_j are Jacobi polynomials. Using the formulas of Table 2, and the new variables

$$\begin{aligned} s &= x_2 + x_1, & x_1 &= s(1-r)/2, \\ r &= (x_2 - x_1)/(x_2 + x_1); & x_2 &= s(1+r)/2; \end{aligned} \quad (4.2)$$

this equation becomes

$$v^{(k_1)}(s(1-r)/2, z_1) v^{(k_2)}(s(1+r)/2, z_2) = \sum_{j=0}^{\infty} C_1 (-1)^j s^j P_j^{(2k_1-1, 2k_2-1)}(r) v^{(k_1 k_2) k_1 + k_2 + j}(s, z_1, z_2). \quad (4.3)$$

Now we can integrate with respect to the weight function of Jacobi polynomials :

$$\begin{aligned} \int_{-1}^1 P_j^{(2k_1-1, 2k_2-1)}(r) v^{(k_1)}\left(\frac{s(1-r)}{2}, z_1\right) v^{(k_2)}\left(\frac{s(1+r)}{2}, z_2\right) (1-r)^{2k_1-1} (1+r)^{2k_2-1} dr \\ = C_1 h_j (-1)^j s^j v^{(k_1 k_2) k_1 + k_2 + j}(s, z_1, z_2), \end{aligned} \quad (4.4)$$

where h_j is the norm of the Jacobi polynomials of degree j , i.e.

$$h_j = \frac{2^{2k_1+2k_2-1}}{2k_1+2k_2+2j-1} \frac{\Gamma(2k_1+j)\Gamma(2k_2+j)}{j!\Gamma(2k_1+2k_2+j-1)}. \quad (4.5)$$

Using the expressions for the functions v given in Table 3, we obtain

$$\begin{aligned} C_1 h_j (-1)^j s^j v^{(k_1 k_2) k_1 + k_2 + j}(s, z_1, z_2) &= (1-z_1)^{-2k_1} (1-z_2)^{-2k_2} e^{\frac{s}{2}(\frac{z_1}{z_1-1} + \frac{z_2}{z_2-1})} \\ \times \int_{-1}^1 P_j^{(2k_1-1, 2k_2-1)}(r) e^{\frac{s(z_1-z_2)}{2(z_1-1)(z_2-1)}} (1-r)^{2k_1-1} (1+r)^{2k_2-1} dr. \end{aligned} \quad (4.6)$$

Lemma 1

$$\begin{aligned} I &= \int_{-1}^1 (1-r)^a (1+r)^b P_j^{(a,b)}(r) e^{cr} dr = \\ &= 2^{a+b+1} \frac{\Gamma(a+j+1)\Gamma(b+j+1)}{j!\Gamma(a+b+2j+2)} e^{-c} (2c)^j {}_1F_1 \left[\begin{matrix} b+j+1 \\ a+b+2j+2 \end{matrix}; 2c \right]. \end{aligned} \quad (4.7)$$

Proof. Writing

$$P_j^{(a,b)}(r) = \frac{1}{2^j} \sum_{m=0}^j \binom{j+a}{m} \binom{j+b}{j-m} (r-1)^{j-m} (r+1)^m, \quad (4.8)$$

and using

$$\begin{aligned} \int_{-1}^1 (1-r)^\alpha (1+r)^\beta e^{\gamma r} dr = \\ 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} e^{-\gamma} {}_1F_1 \left[\begin{matrix} \beta+1 \\ \alpha+\beta+2 \end{matrix}; 2\gamma \right], \end{aligned} \quad (4.9)$$

one obtains, after some simplifications,

$$I = (-1)^j 2^{a+b+1} e^{-c} \frac{\Gamma(a+j+1)\Gamma(b+j+1)}{j!\Gamma(a+b+j+2)} \sum_{j=0}^m \frac{(-j)_m}{m!} {}_1F_1 \left[\begin{matrix} b+m+1 \\ a+b+j+2 \end{matrix}; 2c \right]. \quad (4.10)$$

In the sum over m , the ${}_1F_1$ is written explicitly as a series, the two summation variables are changed, and then Vandermonde's summation theorem can be applied; putting $\alpha = a+b+j+2$ and $\beta = b+1$, this gives explicitly :

$$\begin{aligned} \sum_{j=0}^m \frac{(-j)_m}{m!} {}_1F_1 \left[\begin{matrix} \beta+m \\ \alpha \end{matrix}; \gamma \right] &= \sum_{m=0}^j \frac{(-j)_m}{m!} \sum_{k=0}^{\infty} \frac{(\beta+m)_k \gamma^k}{(\alpha)_k k!} \\ &= \sum_{k=0}^{\infty} \frac{(\beta)_k \gamma^k}{(\alpha)_k k!} \sum_{m=0}^j \frac{(b+k)_m (-j)_m}{(b)_m m!} \\ &= \sum_{k=0}^{\infty} \frac{(\beta)_k \gamma^k}{(\alpha)_k k!} \frac{(-k)_j}{(b)_j}. \end{aligned} \quad (4.11)$$

In the last sum k goes from j upto ∞ , so let $l = k - j$, and rewrite the series; one finds

$$\frac{(-1)^j \gamma^j}{(\alpha)_j} \sum_{l=0}^{\infty} \frac{(\beta+j)_l \gamma^l}{(\alpha+j)_l l!} = \frac{(-1)^j \gamma^j}{(\alpha)_j} {}_1F_1 \left[\begin{matrix} \beta+j \\ \alpha+j \end{matrix}; \gamma \right]. \quad (4.12)$$

Putting this back in (4.10) finally proves the lemma.

As a consequence, we now have the following explicit form for $v^{(k_1 k_2)k_1+k_2+j}(s, z_1, z_2)$:

$$\begin{aligned} v^{(k_1 k_2)k_1+k_2+j}(s, z_1, z_2) &= (-1)^j \left[\frac{(2k_1)_j (2k_2)_j}{j! (2k_1 + 2k_2 + j - 1)_j} \right]^{1/2} (1 - z_1)^{-2k_1-j} (1 - z_2)^{-2k_2-j} \\ &\times (z_1 - z_2)^j e^{s z_1 / (z_1 - 1)} {}_1F_1 \left[\begin{matrix} 2k_1 + j \\ 2k_1 + 2k_2 + 2j \end{matrix}; \frac{s(z_1 - z_2)}{(z_1 - 1)(z_2 - 1)} \right]. \end{aligned} \quad (4.13)$$

On the other hand, we know from (2.12) that ($k = k_1 + k_2 + j$) :

$$v^{(k_1 k_2)k}(s, z_1, z_2) = \sum_{n=0}^{\infty} l_n^{(k)}(s) e_n^{(k_1 k_2)k}(z_1, z_2), \quad (4.14)$$

with $l_n^{(k)}(s)$ given in Table 1 and $e_n^{(k_1 k_2)k}(z_1, z_2)$ given in (3.6). Thus we obtain,

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(2k)}(s) z_1^n {}_2F_1 \left[\begin{matrix} -n, 2k_2 + j \\ 2k_1 + 2k_2 + 2j \end{matrix}; 1 - z_2/z_1 \right] &= \\ (1 - z_1)^{-2k_1-j} (1 - z_2)^{-2k_2-j} e^{s z_1 / (z_1 - 1)} {}_1F_1 \left[\begin{matrix} 2k_1 + j \\ 2k_1 + 2k_2 + 2j \end{matrix}; \frac{s(z_1 - z_2)}{(z_1 - 1)(z_2 - 1)} \right]. \end{aligned} \quad (4.15)$$

This can be rewritten in a more appropriate form :

Proposition 2

$$\sum_{n=0}^{\infty} L_n^{(b-1)}(s) z_1^n {}_2F_1 \left[\begin{matrix} -n, a \\ b \end{matrix}; 1 - \frac{z_2}{z_1} \right] = (1 - z_1)^{a-b} (1 - z_2)^{-a} e^{sz_1/(z_1-1)} {}_1F_1 \left[\begin{matrix} a \\ b \end{matrix}; \frac{s(z_1 - z_2)}{(z_1 - 1)(z_2 - 1)} \right]. \quad (4.16)$$

This is a generalization of two classical generating functions for the Laguerre polynomials. For $a = 0$, or $z_1 = z_2$, it reduces to the first classical result

$$\sum_{n=0}^{\infty} L_n^{(b-1)}(s) z_1^n = (1 - z_1)^{-b} e^{sz_1/(z_1-1)}; \quad (4.17)$$

for $z_2 = 0$, it reduces to

$$\sum_{n=0}^{\infty} L_n^{(b-1)}(s) \frac{(b-a)_n}{(b)_n} z_1^n = (1 - z_1)^{-b} {}_1F_1 \left[\begin{matrix} b-a \\ b \end{matrix}; \frac{sz_1}{(z_1 - 1)} \right]. \quad (4.18)$$

Note that (4.16) can be written in a more general form :

$$\sum_{n=0}^{\infty} {}_1F_1 \left[\begin{matrix} -n \\ b \end{matrix}; x \right] {}_2F_1 \left[\begin{matrix} -n, a \\ b \end{matrix}; y \right] \frac{(b)_n}{n!} z^n = (1 - z)^{a-b} (1 - z + yz)^{-a} e^{xz/(z-1)} {}_1F_1 \left[\begin{matrix} a \\ b \end{matrix}; \frac{xyz}{(1-z)(1-z+yz)} \right]. \quad (4.19)$$

If one is not interested in the particular form of $v^{(k_1 k_2)k_1+k_2+j}(s, z_1, z_2)$, but only in comparing the two expressions of this vector in order to obtain (4.15), many simplifications take place. In particular, note that (4.15) does not explicitly depend upon j but only upon $2k_1 + j$ and $2k_2 + j$. Thus, one can obtain the same formula (4.16) just by working out only the vector with $j = 0$. This will be done for the remaining cases in the following sections.

V The case of Meixner and Meixner-Pollaczek polynomials

Consider the case $X = X_c$; then (4.1) holds with S_j proportional to a Hahn polynomial. First, assume that $s = x_1 + x_2$ is a positive integer (later this condition will disappear by analytic continuation). We can then apply the (discrete) orthogonality relation for Hahn polynomials, and obtain the $j = 0$ component

$$v^{(k_1 k_2)k_1+k_2}(s, z_1, z_2) \frac{(2k+1+2k_2-1)_{s+1}}{s!(2k_1+2k_2-1)} = \sum_{x_1=0}^s \frac{(2k_1)_{x_1} (2k_2)_{s-x_1}}{x_1! (s-x_1)!} v^{(k_1)}(x_1, z_1) v^{(k_2)}(s-x_1, z_2). \quad (5.1)$$

With the explicit forms for $v^{(k_1)}$ and $v^{(k_2)}$ from Table 3, this gives rise to

$$v^{(k_1 k_2)k_1+k_2}(s, z_1, z_2) = (1 - cz_1)^{-2k_1} (1 - z_2/c)^s (1 - cz_2)^{-s-2k_2} {}_2F_1 \left[\begin{matrix} -s, 2k_1 \\ 2k_1 + 2k_2 \end{matrix}; 1 - \frac{z_2}{z_1} \right]. \quad (5.2)$$

On the other hand, (4.14) now becomes

$$v^{(k_1 k_2)k_1+k_2}(s, z_1, z_2) = \sum_{n=0}^{\infty} \frac{(2k_1 + 2k_2)_n}{n!} M_n(s; 2k_1 + 2k_2, c^2) c^n z_1^n {}_2F_1 \left[\begin{matrix} -n, 2k_2 \\ 2k_1 + 2k_2 \end{matrix}; 1 - \frac{z_2}{z_1} \right]. \quad (5.3)$$

Applying a series transformation on the last ${}_2F_1$ yields

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(b)_n}{n!} M_n(s; b, c^2) c^n z_2^n {}_2F_1 \left[\begin{matrix} -n, a \\ b \end{matrix}; 1 - \frac{z_2}{z_1} \right] = \\ (1 - cz_1)^{-a} (1 - cz_2)^{a-b} \left(\frac{1 - z_2/c}{1 - cz_2} \right)^s {}_2F_1 \left[\begin{matrix} -s, a \\ b \end{matrix}; \frac{(c - 1/c)(z_2 - z_1)}{(1 - cz_1)(1 - z_2/c)} \right]. \end{aligned} \quad (5.4)$$

Given the Meixner polynomials in terms of a ${}_2F_1$, this leads to the following series expression :

Proposition 3

$$\begin{aligned} \sum_{n=0}^{\infty} {}_2F_1 \left[\begin{matrix} -n, a \\ c \end{matrix}; x \right] {}_2F_1 \left[\begin{matrix} -n, b \\ c \end{matrix}; y \right] \frac{(c)_n z^n}{n!} = \\ (1 - z + xz)^{-a} (1 - z + yz)^{-b} (1 - z)^{a+b-c} {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix}; \frac{xyz}{(1 - z + xz)(1 - z + yz)} \right]. \end{aligned} \quad (5.5)$$

This formula was already given in Ref. [14], p. 85, eq. (12), and can be interpreted as the Poisson kernel for Meixner or Meixner-Pollaczek polynomials [15]. Observe that (4.19) is a limiting case of (5.5) : putting $x = x'/a$ in (5.5) and taking the limit $a \rightarrow \infty$ leads to (4.19).

The case of Meixner-Pollaczek polynomials leads essentially to the same formulas.

VI The algebra $U_q(su(1, 1))$

Let $U_q(sl(2, \mathbb{C}))$ ($0 < q < 1$) be the complex unital associative algebra generated by A, B, C, D subject to the relations[§]

$$AD = 1 = DA, \quad AB = q^{1/2}BA, \quad AC = q^{-1/2}CA, \quad BC - CB = \frac{A^2 - D^2}{q^{1/2} - q^{-1/2}}. \quad (6.1)$$

This algebra can be equipped with a comultiplication, counit, and antipode, turning it into a Hopf algebra [16, 8]. The Hopf $*$ -algebra $U_q(su(1, 1))$ has the following $*$ -structure :

$$A^* = A, \quad B^* = -C, \quad C^* = -B, \quad D^* = D. \quad (6.2)$$

The positive discrete representations are labeled by a positive real number k . The representation space is again $\ell^2(\mathbb{Z}_+)$, with orthonormal basis vectors denoted by $e_n^{(k)}$, with

[§]Compared to Ref. [6], q is replaced by $q^{1/2}$ in order to have q as basis of the basic hypergeometric series appearing later.

$n = 0, 1, 2, \dots$. The explicit action of the generators in this representation (k) is given by [6]

$$A e_n^{(k)} = q^{(k+n)/2} e_n^{(k)}, \quad (6.3)$$

$$C e_n^{(k)} = q^{(1-2k-2n)/4} \frac{\sqrt{(1-q^n)(1-q^{2k+n-1})}}{q^{1/2} - q^{-1/2}} e_{n-1}^{(k)}, \quad (6.4)$$

$$B e_n^{(k)} = q^{-(1+2k+2n)/4} \frac{\sqrt{(1-q^{n+1})(1-q^{2k+n})}}{q^{-1/2} - q^{1/2}} e_{n+1}^{(k)}. \quad (6.5)$$

Let $s \in \mathbb{R} \setminus \{0\}$, and define

$$Y_s = q^{1/4} B - q^{-1/4} C + \frac{s^{-1} + s}{q^{-1/2} - q^{1/2}} (A - D). \quad (6.6)$$

Under the comultiplication

$$\begin{aligned} \Delta(A) &= A \otimes A, & \Delta(B) &= A \otimes B + B \otimes D, \\ \Delta(C) &= A \otimes C + C \otimes D, & \Delta(D) &= D \otimes D, \end{aligned} \quad (6.7)$$

Y_s is twisted primitive, and $Y_s A$ is a self-adjoint element. Here, $Y_s A$ is playing the role of the operator X for $su(1, 1)$. In a previous paper [6], the formal eigenvectors

$$v^{(k)}(x) = \sum_{n=0}^{\infty} l_n^{(k)}(x) e_n^{(k)} \quad (6.8)$$

of $Y_s A$ have been obtained. In order to express the results of [6], recall the definition of Askey-Wilson [17] and Al-Salam and Chihara polynomials [18]. The Askey-Wilson polynomial is defined by ($x = \cos \theta$) :

$$p_m(x; a, b, c, d|q) = a^{-m} (ab, ac, ad; q)_m {}_4\varphi_3 \left[\begin{matrix} q^{-m}, abcdq^{m-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q \right] \quad (6.9)$$

and it is symmetric in its parameters a, b, c and d . In (6.9), the notation for basic hypergeometric series and for the shifted q -factorial is taken from [12]. The Al-Salam and Chihara polynomials are obtained by taking $c = d = 0$ in the Askey-Wilson polynomials;

$$s_m(x; a, b|q) = p_m(x; a, b, 0, 0|q) = a^{-m} (ab; q)_m {}_3\varphi_2 \left[\begin{matrix} q^{-m}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix}; q, q \right]. \quad (6.10)$$

Using the abbreviation $\mu(x) = (x + x^{-1})/2$, it was shown in [6] that for

$$l_n^{(k)}(x) = \frac{1}{\sqrt{(q, q^{2k}; q)_n}} s_n(\mu(x); q^k s, q^k/s|q), \quad (6.11)$$

the vectors (6.8) are formal eigenvectors of $Y_s A$ for the eigenvalue

$$\lambda(x) = 2(\mu(s) - \mu(x))/(q^{1/2} - q^{-1/2}).$$

The tensor product of two $U_q(su(1, 1))$ representations $(k_1) \otimes (k_2)$ is the same as in (2.10), and the Clebsch-Gordan coefficients in

$$e_n^{(k_1 k_2)k} = \sum_{n_1, n_2} C_{n_1, n_2, n}^{k_1, k_2, k} e_{n_1}^{(k_1)} \otimes e_{n_2}^{(k_2)}, \quad (6.12)$$

have been determined, e.g., in Ref. [6]. In the tensor product space, the generalized eigenvectors of $\Delta(Y_s A)$ have been considered, both in coupled and uncoupled form. It was shown that

$$v^{k_1; k_2}(x_1, x_2) = \sum_{n_1, n_2} \frac{s_{n_1}(\mu(x_1); q^{k_1} x_2, q^{k_1}/x_2 | q)}{\sqrt{(q, q^{2k_1}; q)_{n_1}}} \frac{s_{n_2}(\mu(x_2); q^{k_2} s, q^{k_2}/s | q)}{\sqrt{(q, q^{2k_2}; q)_{n_2}}} e_{n_1}^{(k_1)} \otimes e_{n_2}^{(k_2)} \quad (6.13)$$

is a generalized eigenvector of $\Delta(Y_s A)$ for the eigenvalue $\lambda(x_1)$. In “coupled” form,

$$v^{(k_1 k_2)k}(x_1) = \sum_{n=0}^{\infty} \frac{s_n(\mu(x_1); q^k s, q^k/s | q)}{\sqrt{(q, q^{2k}; q)_n}} e_n^{(k_1 k_2)k}, \quad (6.14)$$

is a generalized eigenvector of $\Delta(Y_s A)$ for the same eigenvalue. The relation between the two generalized eigenvectors was given in [6] :

$$v^{k_1; k_2}(x_1, x_2) = \sum_{j=0}^{\infty} C_j p_j(\mu(x_2); q^{k_1} x_1, q^{k_1}/x_1, q^{k_2} s, q^{k_2}/s | q) v^{(k_1 k_2)k_1 + k_2 + j}(x_1), \quad (6.15)$$

where p_j is an Askey-Wilson polynomial and

$$C_j = ((q, q^{2k_1}, q^{2k_2}, q^{2k_1+2k_2+j-1}; q)_j)^{-1/2}. \quad (6.16)$$

VII Realization of $U_q(su(1, 1))$

A realization of $U_q(su(1, 1))$ and the representation labelled by (k) is closely related to the one given for $su(1, 1)$ in Section III. The Hilbert space is the same (up to a rescaling of the inner product), and the basis functions are now

$$e_n^{(k)} \equiv e_n^{(k)}(z) = \sqrt{\frac{(q^{2k}; q)_n}{(q; q)_n}} z^n. \quad (7.1)$$

The realization of the $U_q(su(1, 1))$ generators is given in terms of the operators

$$T_q f(z) = f(qz), \quad D_q = \frac{1 - T_q}{(1 - q)z}, \quad (7.2)$$

and reads

$$A = q^{k/2} T_{q^{1/2}}, \quad D = q^{-k/2} T_{q^{-1/2}}, \quad (7.3)$$

$$B = q^{(1-2k)/4} \left(z^2 D_q T_{q^{-1/2}} + \frac{1 - q^{2k}}{1 - q} z T_{q^{1/2}} \right), \quad (7.4)$$

$$C = -q^{(3-2k)/4} D_q T_{q^{-1/2}}. \quad (7.5)$$

For this realization, the formal vectors (6.8) become

$$v^{(k)}(x, z) = \sum_{n=0}^{\infty} s_n(\mu(x); q^k s, q^k/s|q) \frac{z^n}{(q; q)_n} \quad (7.6)$$

$$= \frac{(q^k z s, q^k z/s; q)_{\infty}}{(zx, z/x; q)_{\infty}}, \quad (7.7)$$

following from a known generating function for the Al-Salam and Chihara polynomials [10]. We also wish to obtain an explicit form for the coupled vectors $e_n^{(k_1 k_2)k}(z_1, z_2)$. Using the proof of Lemma 4.4 of [6], one finds ($k = k_1 + k_2 + j$) :

$$\begin{aligned} e_n^{(k_1 k_2)k}(z_1, z_2) &= q^{-nj-nk_1} z_2^{n+j} (q^{k_1} z_1/z_2; q)_j \sqrt{\frac{(q^{2k_1}, q^{2k_2}; q)_j}{(q, q^{2k_1+2k_2+j-1}; q)_j}} \\ &\times \sqrt{\frac{(q^{2k_1+2k_2+2j}; q)_n}{(q; q)_n}} {}_3\varphi_2 \left[\begin{matrix} q^{-n}, q^{2k_1+j}, q^{k_1+j} z_1/z_2 \\ q^{2k_1+2k_2+2j}, 0 \end{matrix}; q, q \right]. \end{aligned} \quad (7.8)$$

VIII Poisson kernel for Al-Salam–Chihara polynomials

In the realization of the previous section, the formal vectors defined in (6.13) can be rewritten using (7.7) and they become

$$v^{k_1; k_2}(x_1, x_2, z_1, z_2) = \frac{(q^{k_1} z_1 x_2, q^{k_1} z_1/x_2, q^{k_2} z_2 s, q^{k_2} z_2/s; q)_{\infty}}{(z_1 x_1, z_1/x_1, z_2 x_2, z_2/x_2; q)_{\infty}}. \quad (8.1)$$

Note that in all the previous formulas the variables of the Al-Salam and Chihara (or Askey-Wilson) polynomials are $\mu(x_1)$ or $\mu(x_2)$; we can represent $\mu(x_l) = \cos \theta_l$ and thus $x_l = e^{i\theta_l}$ for some real θ_l ($l = 1, 2$). The purpose is now to integrate equation (6.15) so as to obtain an explicit form for $v^{(k_1 k_2)k_1+k_2+j}(x_1, z_1, z_2)$ for $j = 0$. Let

$$(a, b, c, d) = (q^{k_1} x_1, q^{k_1}/x_1, q^{k_2} s, q^{k_2}/s). \quad (8.2)$$

If we assume that $q^{k_2} < |s| < q^{-k_2}$, then $\max\{|a|, |b|, |c|, |d|\} < 1$, and the Askey-Wilson polynomial in (6.15) has an absolute continuous measure. Using the notation

$$h(x; a) = h(x; a; q) = (ae^{i\theta}, ae^{-i\theta}; q)_{\infty}, \quad \text{for } x = \cos \theta, \quad (8.3)$$

and

$$h(x; a, b, c, d) = h(x; a)h(x; b)h(x; c)h(x; d), \quad (8.4)$$

the weight function for the Askey-Wilson polynomials reads [12, §7.5]

$$w(x) \equiv w(x; a, b, c, d) = \frac{h(x; 1, -1, q^{1/2}, q^{-1/2})}{\sqrt{1-x^2} h(x; a, b, c, d)}. \quad (8.5)$$

The orthogonality relation for $p_n(x) \equiv p_n(x; a, b, c, d|q)$ is

$$\int_{-1}^1 p_m(x) p_n(x) w(x) dx = \frac{\delta_{m,n}}{h_n}, \quad (8.6)$$

where

$$h_n = h_0 \frac{(abcdq^{-1}; q)_n (1 - abcdq^{2n-1})}{(1 - abcdq^{-1})(q, ab, ac, ad, bc, bd, cd; q)_n}, \quad (8.7)$$

and

$$h_0 = \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{2\pi(abcd; q)_\infty}. \quad (8.8)$$

It thus follows from (6.15) that

$$v^{(k_1 k_2)k_1+k_2}(x_1, z_1, z_2) = h_0 \int_{-1}^1 w(\mu(x_2); a, b, c, d) v^{k_1; k_2}(x_1, x_2, z_1, z_2) d\mu(x_2). \quad (8.9)$$

Using the explicit form (8.1), the notation

$$f = z_2, \quad g = q^{k_1} z_1, \quad (8.10)$$

and writing $\mu(x_2) = t$, (8.9) becomes

$$v^{(k_1 k_2)k_1+k_2}(x_1, z_1, z_2) = h_0 \frac{(cf, df; q)_\infty}{(g/a, g/b; q)_\infty} \int_{-1}^1 \frac{h(t; g)}{h(t; f)} w(t; a, b, c, d) dt. \quad (8.11)$$

The last integral is a known one; in Ref. [12] it is denoted by $J(a, b, c, d, f, g)$, and shown to be equal to a ${}_8W_7$ series, i.e. a very-well-poised ${}_8\varphi_7$ series. Using eq. (6.3.8) of [12], we obtain

$$\begin{aligned} v^{(k_1 k_2)k_1+k_2}(x_1, z_1, z_2) &= \frac{(ag, bg, cg, df, abcf; q)_\infty}{(af, bf, g/a, g/b, abcg; q)_\infty} \\ &\times {}_8W_7(abcgq^{-1}; ab, ac, bc, g/d, g/f; q, df). \end{aligned} \quad (8.12)$$

Let us now consider the other form of $v^{(k_1 k_2)k_1+k_2}(x_1, z_1, z_2)$, given by (6.14). Using (7.8) and definition (6.10), one finds :

$$\begin{aligned} v^{(k_1 k_2)k_1+k_2}(x_1, z_1, z_2) &= \sum_{n=0}^{\infty} \frac{s_n(\mu(x_1); q^{k_1+k_2}s, q^{k_1+k_2}/s|q)}{\sqrt{(q, q^{2k_1+2k_2}; q)_n}} e_n^{(k_1 k_2)k_1+k_2}(z_1, z_2) \\ &= \sum_{n=0}^{\infty} \frac{(q^{2k_1+2k_2}; q)_n}{(q; q)_n} (q^{-2k_1-k_2} z_2/s)^n {}_3\varphi_2 \left[\begin{matrix} q^{-n}, q^{k_1+k_2} s x_1, q^{k_1+k_2} s/x_1 \\ q^{2k_1+2k_2}, 0 \end{matrix}; q, q \right] \\ &\quad \times {}_3\varphi_2 \left[\begin{matrix} q^{-n}, q^{2k_1}, q^{k_1} z_1/z_2 \\ q^{2k_1+2k_2}, 0 \end{matrix}; q, q \right], \end{aligned} \quad (8.13)$$

or, putting all this in the notation of (8.2) and (8.10),

$$\begin{aligned} v^{(k_1 k_2)k_1+k_2}(x_1, z_1, z_2) &= \\ &\sum_{n=0}^{\infty} \frac{(abcd; q)_n}{(q; q)_n} \left(\frac{f}{abc} \right)^n {}_3\varphi_2 \left[\begin{matrix} q^{-n}, ac, ad \\ abcd, 0 \end{matrix}; q, q \right] {}_3\varphi_2 \left[\begin{matrix} q^{-n}, ab, g/f \\ abcd, 0 \end{matrix}; q, q \right]. \end{aligned} \quad (8.14)$$

Equating (8.14) and (8.12) gives the q -analog of (5.5). Relabelling all the variables, it can be written in the following form :

Proposition 4

$$\sum_{n=0}^{\infty} {}_3\varphi_2 \left[\begin{matrix} q^{-n}, a, b \\ f, 0 \end{matrix}; q, q \right] {}_3\varphi_2 \left[\begin{matrix} q^{-n}, c, d \\ f, 0 \end{matrix}; q, q \right] \frac{(f; q)_n}{(q; q)_n} z^n = \frac{(abcz, abdz, acdz, bcdz, f; q)_{\infty}}{(acz, bcz, adz, bdz, abcdz; q)_{\infty}} {}_8W_7(abcdzq^{-1}; a, b, c, d, abcdz/f; q, fz). \quad (8.15)$$

This formula is the (symmetric) Poisson kernel for Al-Salam–Chihara polynomials, and was derived earlier by classical methods in [19, (14.8)] and [20]. As far as we know, this is the first group theoretical derivation of it.

IX Other $su(1, 1)$ realizations and their consequences

In this section two different realizations of $su(1, 1)$ and its positive discrete series representations will be considered. For the realization considered so far, the typical feature is that the basis functions $e_n^{(k)}$ are simply monomials z^n . Here, the basis functions will have a more complicated form, and the q -analog will not be treated.

Before turning to the realization, it is useful to observe some realization-independent facts. In section II, the operator X_c ($0 < c < 1$) was introduced, and from Table 1 one can see that its spectrum is identical to that of J_0 apart from the factor $(c - 1/c)$. So one can expect that X_c and J_0 are related through a unitary transformation, and this is indeed the case. For $0 < c < 1$, let $\alpha > 0$ be defined through

$$e^{\alpha} = \frac{1+c}{1-c}, \quad (9.1)$$

or inversely,

$$c = \frac{e^{\alpha} - 1}{e^{\alpha} + 1}. \quad (9.2)$$

Defining, as usual,

$$J_1 = \frac{1}{2}(J_+ + J_-), \quad J_2 = \frac{1}{2i}(J_+ - J_-) \quad (9.3)$$

in terms of the $su(1, 1)$ basis (2.1), it is well known that the following identity holds [1, eq. (97)][¶]:

$$\exp(i\alpha J_2) J_0 \exp(-i\alpha J_2) = (\cosh \alpha) J_0 - (\sinh \alpha) J_1. \quad (9.4)$$

Then it follows from (2.6) and the above relation between c and α that

$$X_c = (c - 1/c) \exp(i\alpha J_2) J_0 \exp(-i\alpha J_2). \quad (9.5)$$

Acting with this equation on the formal eigenvectors $v^{(k)}(m)$ of X_c yields

$$J_0 \exp(-i\alpha J_2) v^{(k)}(m) = (k + m) \exp(-i\alpha J_2) v^{(k)}(m), \quad (9.6)$$

[¶]This equation can also be obtained by applying the “unitary trick” $J_x \rightarrow iJ_1$, $J_y \rightarrow iJ_2$, $J_z \rightarrow J_0$, $\theta \rightarrow i\alpha$ to the $su(2)$ relation $\exp(-i\theta J_y) J_z \exp(i\theta J_y) = (\sin \theta) J_x + (\cos \theta) J_z$, see [21, eq. (D.10)].

thus up to a normalization factor N_m these elements must coincide with the eigenvectors $e_m^{(k)}$ of J_0 , leading to

$$\exp(i\alpha J_2) e_m^{(k)} = N_m v^{(k)}(m). \quad (9.7)$$

In other words, using (2.3), Table 1, and the orthogonality for Meixner polynomials, this yields

$$\exp(i\alpha J_2) e_m^{(k)} = \phi_m \sum_{n=0}^{\infty} (1-c^2)^k c^{m+n} \sqrt{\frac{(2k)_m (2k)_n}{m! n!}} M_n(m; 2k; c^2) e_n^{(k)}, \quad (9.8)$$

where ϕ_m is a phase factor ($|\phi_m| = 1$). This relation is realization-independent, and for particular realizations it leads to interesting identities. First it should be noted that for the realization of section III, the explicit form of $\exp(i\alpha J_2) e_m^{(k)}$ becomes rather complicated and (9.8) does not reduce to a simple relation. Here, we shall give two different realizations for which (9.8) implies an interesting relation.

First, let $w \geq 1$, $r \in (0, \infty)$ and $p_r = -i \left(\frac{d}{dr} + \frac{1}{r} \right)$. In [1, eq. (75)], the following realization of $su(1, 1)$ is considered :

$$\begin{aligned} J_1 &= \frac{1}{2} (w^{-2} r^{2-w} p_r^2 + \xi r^{-w} - r^w), \\ J_2 &= w^{-1} (r p_r - i(w-1)/2), \\ J_0 &= \frac{1}{2} (w^{-2} r^{2-w} p_r^2 + \xi r^{-w} + r^w). \end{aligned} \quad (9.9)$$

These provide a realization of the $su(1, 1)$ basis which are Hermitian under the scalar product

$$\langle f, g \rangle = \int_0^\infty f^*(r) g(r) r^w dr. \quad (9.10)$$

More precisely, $\mathcal{D}^+(k)$ consist of functions of r that are integrable with respect to the above measure, and (9.9) forms a realization of the $su(1, 1)$ basis in this representation $\mathcal{D}^+(k)$ provided

$$\xi = k(k-1) - W(W-1), \quad \text{where} \quad W = \frac{w+1}{2w}. \quad (9.11)$$

The normalized eigenfunctions $e_m^{(k)}$ of J_0 are given by

$$e_m^{(k)} = 2^W \sqrt{\frac{w m!}{\Gamma(2k+m)}} \exp(-r^w) (2r^w)^{k-W} L_m^{(2k-1)}(2r^w), \quad (9.12)$$

in terms of the Laguerre polynomials (2.7), see [1, eq. (89)]. An advantage of this realization is that J_2 is a linear combination of $r \frac{d}{dr}$, and hence $\exp(i\alpha J_2)$ acting on a function of r takes a simple form, i.e.

$$\exp(i\alpha J_2) f(r) = e^{\alpha W} f(e^{\alpha/w} r). \quad (9.13)$$

Using all this information, (9.8) becomes

$$\frac{m!}{(2k)_m} \exp(-e^\alpha r^w) e^{\alpha k} L_m^{(2k-1)}(2e^\alpha r^w) = \quad (9.14)$$

$$\phi_m \sum_{n=0}^{\infty} (1-c^2)^k c^{n+m} M_n(m; 2k; c^2) \exp(-r^w) L_n^{(2k-1)}(2r^w). \quad (9.15)$$

Making the replacements

$$x = 2r^w, \quad \rho = e^\alpha > 1, \quad a = 2k - 1, \quad (9.16)$$

and keeping in mind the relation (9.2), (9.15) takes a simpler form. In fact, rather than leaving this as an infinite series expression in terms of Laguerre polynomials, it is more convenient to integrate with respect to the orthogonality measure of Laguerre polynomials and then obtain the equivalent expression :

Corollary 5 For $\rho > 1$ and $a > -1$,

$$\begin{aligned} & \int_0^\infty L_m^{(a)}(\rho x) L_n^{(a)}(x) \exp(-(\rho + 1)x/2) x^a dx = \\ & (-1)^m \frac{\Gamma(a + n + 1)}{n!} \frac{(a + 1)_m}{m!} \left(\frac{2}{\rho + 1} \right)^{a+1} \left(\frac{\rho - 1}{\rho + 1} \right)^{n+m} M_n(m; a + 1; \left(\frac{\rho - 1}{\rho + 1} \right)^2). \end{aligned} \quad (9.17)$$

The fact that the phase factor ϕ_m is equal to $(-1)^m$ can easily be derived from the case $n = 0$. Although we have not found (9.17) in the literature, it is perhaps not new, and it can probably be derived by classical methods as well. Nevertheless, in our treatment it does not require any extra work to derive it, and the appearance of a Meixner polynomial in the rhs of (9.17) has a natural explanation.

As a second example of a different realization, we shall consider the well known boson realization of $su(1, 1)$, see also [1]. In terms of the annihilation and creation operators

$$a = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right), \quad (9.18)$$

we have the realization

$$\begin{aligned} J_1 &= \frac{1}{4} \left((a^\dagger)^2 + a^2 \right), \\ J_2 &= \frac{1}{4i} \left((a^\dagger)^2 - a^2 \right), \\ J_0 &= \frac{1}{4} \left(aa^\dagger + a^\dagger a \right). \end{aligned} \quad (9.19)$$

The representation space is $L_2(\mathbb{R})$, with inner product $\langle f, g \rangle = \int f^*(x)g(x)dx$. This corresponds to the positive discrete series representation $\mathcal{D}^+(1/4)$, so $k = 1/4$. The normalized eigenvectors $e_n^{(1/4)}$ are given by

$$e_n^{(1/4)} = \frac{1}{\pi^{1/4} 2^n \sqrt{(2n)!}} \exp(-x^2/2) H_{2n}(x), \quad (9.20)$$

with $H_n(x)$ the usual notation for Hermite polynomials [22, §22]. Since also for this realization J_2 is a linear combination of $x \frac{d}{dx}$, the simple relation

$$\exp(i\alpha J_2) f(x) = e^{-\alpha/4} f(e^{-\alpha/2} x) \quad (9.21)$$

holds. Plugging all this information in (9.8) gives an expression similar to (9.15), but with Hermite polynomials instead of Laguerre polynomials. Replacing herein $e^{-\alpha/2}$ by λ , and going to the equivalent form in terms of an integral, this result can be expressed as follows :

Corollary 6

$$\int_{-\infty}^{\infty} H_{2m}(\lambda x) H_{2n}(x) \exp(-(\lambda^2 + 1)x^2/2) dx = \quad (9.22)$$

$$(-1)^m \sqrt{\frac{2\pi}{1+\lambda^2}} \left(\frac{1-\lambda^2}{1+\lambda^2} \right)^{n+m} \frac{(2m)!(2n)!}{m!n!} M_n(m; 1/2; \left(\frac{1-\lambda^2}{1+\lambda^2} \right)^2).$$

The phase factor ϕ_m is again easily derived. Again we have not found (9.22) in the literature. Although it can be derived by classical methods, here it has required no extra work.

With the two different realizations considered in this section, the analysis can be continued. For example, another interesting equation is the realization of (2.13) in terms of the above Laguerre polynomials.

In the case of the quantum algebra $U_q(su(1,1))$, similar developments can be made provided one can find sufficiently simple realizations in terms of q -difference operators and q -special functions.

To conclude, using the explicit knowledge of the expansion coefficients of the eigenstates of a general Hamiltonian in terms of the J_0 eigenstates, we have deduced a number of identities for orthogonal polynomials. By choosing a realization for which the J_0 eigenstates coincide with monomials z^n , generating functions or Poisson kernels were derived. Considering the q -analog of this realization, the Poisson kernel for Al-Salam–Chihara polynomials is found. Finally, we have shown that different realizations for the $su(1,1)$ case give rise to explicit formulas for integrals over orthogonal polynomials expressed as a Meixner polynomial.

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